4. P-C PRESENTATIONS

§ 4.1. Definition and Examples

The presentation of a metacyclic group has relations that express a power of the second generator as a word in the first and commutator relators. We now generalise this.

A **power-commutator presentation** (**PCP**) of a group is one of the form:

$$\langle A_1, .., A_k \mid A_i^{n_i} = P_i \text{ for } i = 1, .., k \text{ and } [A_i, A_j] = C_{ij} \text{ for } 0 \le i < j \le k \rangle$$

where each P_i is a word in the generators up to A_{i-1} (with $P_1 = 1$) and each C_{ij} is a word in the generators up to A_{j-1}.

We can write every element of such a group in the form: $A_k^{a_k} A_{k-1}^{a_{k-1}} \ldots A_1^{a_1}$ where $0 \le a_i < n_i$ for each i.

The reason for the reverse order is because $A_iA_j = A_jA_iC_{ij}$ for all $i < j$ and so writing the generators in reverse order is more convenient.

Given a word in the generators we bring all the A_k's to the left using the relations $A_iA_j = A_jA_iC_{ij}$. The word C_{ij} involves only the generators up to A_{j-1}. We now bring all the A_{j-1}'s to the left, immediately to the right of the A_j's. We continue in this way.

The order of such a group is at most $n_1n_2 \ldots n_k$, but it can be smaller. Let us consider the special cases where the number of generators is small.

One Generator: A PCP with one generator will have the form $G = \langle A \mid A^n = 1\rangle$ and so the group will be a finite cyclic group. Clearly $G' = 1$ in this case.

Two Generators: A PCP with two generators will have the form:

$$G = \langle A, B \mid A^m = 1, B^n = A^k, [A, B] = A^r\rangle$$

and so the group will be a finite metacyclic group. Clearly $G'' = 1$ in this case.

Three Generators: A PCP with three generators will have the form:

$$\langle A, B, C \mid A^m = 1, B^n = A^k , C^d = B^eA^f, [A, B] = A^r, [A, C] = B^hA^k, [B, C] = B^uA^v\rangle.$$

The subgroup $H = \langle A, B\rangle$ will be normal, since

$$C^{-1}AC = AB^hA^k \text{ and } C^{-1}BC = B^{u+1}A^v.$$

H will contain a normal subgroup $K = \langle A\rangle$, which may not be normal in G. Since G/H, H/K and K are cyclic,

$G''' = 1$. The soluble length of G is thus at most 3 (but it could be less).

Example 1: Let G be the group

$$\langle A, B, C, D \mid A^4, B^2 = A^2, C^2 = A^2, D^3, [B, C] = A^2, [B, D] = CBA^2, [C, D] = BA^2 \rangle.$$

[Here, to save space, we omit commutators where generators commute.]

Write the product $(DCBA)(D^2CA^3)$ in the form $D^pC^qB^rA^s$ where $0 \le p < 3$, $0 \le q < 2$, $0 \le r < 2$ and $0 \le s < 4$.

Solution: From the power relations we have:
(P1) $A^4 = 1$;
(P2) $B^2 = A^2$;
(P3) $C^2 = A^2$;
(P4) $D^3 = 1$.
From the commutator relations we have:
(C1) $AB = BA$; $AC = CA$; $AD = DA$;
(C2) $BC = CBA^2$;
(C3) $BD = DBCBA^2 = DCBA^2BA^2 = DCB^2A^4 = DCA^2$;
(C4) $CD = DCBA^2$.
Hence $(DCBA)(D^2CA^3) = (DCBD^2C)A^4$ by C1
$= DC(BD)DC$
$= DC(DCA^2)DC$ by C3
$= (DCDCDC)A^2$ by C1
$= D(CD)(CD)CA^2$
$= D(DCBA^2)(DCBA^2)CA^2$ by C4

$= (D^2CBDCBC)A^6$ by C1
$= D^2CBDCBCA^2$ by P1
$= D^2C(BD)C(BC)A^2$
$= D^2C(DCA^2)C(CBA^2)A^2$
by C3 and C2
$= (D^2CDC^3B)A^6$ by C1
$= (D^2CDC^3B)A^2$ by P1
$= (D^2CD)(C^2)(CBA^2)$
$= (D^2CD)(A^2)(CBA^2)$ by P3
$= (D^2CDCB)A^4$ by C1
$= D^2CDCB$ by P1
$= (D^2)(CD)(CB)$
$= (D^2)(DCBA^2)(CB)$ by C4
$= (D^3CBCB)A^2$ by C1
$= (CBCB)A^2$ by P4
$= C(BC)(BA^2)$
$= C(CBA^2)(BA^2)$ by C2
$= (C^2B^2)A^4$ by C1
$= C^2B^2$ by P1
$= C^2A^2$ by P2.

Example 2: $G = \langle A, B, C \mid A^4, B^4, C^2, [A, C] = B^2\rangle$. This group has order 32 and its elements are of the form $C^iB^jA^k$ where $i, j = 0, 1, 2, 3$ and $k = 0, 1$.
Since $[A, C] = B^2$, $AC = CAB^2$. So every time a C moves to the left across an A, a factor of B^2 is introduced. Since B is in the centre the A's and B's can be brought to the right.

A typical product is:

$$C^iB^jA^k \times C^uB^vA^w = C^{i+u}B^{2uk+j+v}A^{k+w}.$$

(We introduce *uk* factors of B^2 because we have to move a C past an A *uk* times.)

For example:

$$C^3B^3A^3 \times CB^2A^3 = C^3B^3\,(CA^3B^6)\,B^2A^3$$
$$= C^4B^3A^3A^3 = B^3A^2$$

The bottom-right hand portion of the group table is:

	$\mathbf{CB^2A^3}$	$\mathbf{CB^3}$	$\mathbf{CB^3A}$	$\mathbf{CB^3A^2}$	$\mathbf{CB^3A^3}$
....	....	....	....	....	....
$\mathbf{CB^3}$	BA^3	B^2	B^2A	B^2A^2	B^2A^3
$\mathbf{CB^3A}$	B^3	A	A^2	A^3	1
$\mathbf{CB^3A^2}$	BA	B^2A^2	B^2A^3	B^2	B^2A
$\mathbf{CB^3A^3}$	B^3A^2	A^3	1	A	A^2

Example 3:

Let $G = \langle A, B, C \mid A^{60} = 1, B^{10} = A^{12}, C^4 = B^3,$
$[A, B] = A^9, [A, C] = A^{14}, [B, C] = BA\rangle$. What is $|G|$?

Solution: GCD(60, 9) = 3 so $A^3 = 1$.

$\mathrm{GCD}(10, 2^4 - 1) = 5$ so $B^5 \in \langle A\rangle$.

Hence |G| divides 60.

However we can do better than that.

Since $C^{-1}AC = A^{15} = 1$, we have A = 1.

$C^{-1}BC = B^2A = B^2$. Cubing both sides, $C^{-1}B^3C = B^6$.

But $B^3 = C^4$ so $C^{-1}B^3C = B^3$.

Hence $B^6 = B^3$ and so $B^3 = 1$.

However $B^5 \in \langle A\rangle = 1$ so $B^{\mathrm{GCD}(3,\,5)} = 1$ whence B = 1.

So $G = \langle C \mid C^4 \rangle$ and so $|G| = 4$.

§ 4.2. P-C Presentations and Soluble Groups

Theorem 1: The group G has a power-commutator presentation if and only if it is a finite soluble group.

Proof: Suppose G has the presentation:

$$\langle A_1,.., A_k \mid A_i^{n_i} = P_i(A_1,.., A_{i-1}) \text{ for } i = 1,.., k,$$
$$[A_i, A_j] = C_{ij}(A_1,.., A_{j-1}) \text{ for } 0 \le i < j \le k \rangle.$$

For $r = 1, 2, \ldots, k$ let $G_r = \langle A_1, \ldots, A_r \rangle$ be the subgroup generated by the first r generators. Suppose $r > 1$.

For $i < r$, $[A_i, A_r] \in G_{r-1}$ and so it follows that $G_r' \le G_{r-1}$. Also $G_1' = 1$. Hence G is soluble of length at most k.

Conversely suppose G is a finite soluble group. By induction we may suppose that G′ has a power-commutator presentation:

$$\langle A_1,.., A_k \mid A_i^{n_i} = P_i(A_1,.., A_{i-1}) \text{ for } i = 1,.., k,$$
$$[A_i, A_j] = C_{ij}(A_1,.., A_{j-1}) \text{ for } 0 \le i < j \le k \rangle.$$

Now G/G′ is a finite abelian group. Let G/G′ be generated by the cosets $B_1G', \ldots, B_sG'$. Clearly G is generated by $\{A_1, \ldots, A_k, B_1, \ldots, B_s\}$. Now if the coset B_iG' has order m_i then $B_i^{m_i} \in G'$ and so $B_i^{m_i} \in \langle A_1, \ldots, A_k \rangle$.

For $i = 1, \ldots, k$ and $j = 1, \ldots, s$ we have $[A_i, B_j] \in G'$ whence $[A_i, B_j] \in \langle A_1, \ldots, A_k \rangle$.

Similarly for $1 \le i < j \le s$ we have $[B_i, B_j] \in G'$ whence $[B_i, B_j] \in \langle A_1, \ldots, A_k \rangle$.
This leads to a PCP for G. ✋☺

EXERCISES FOR CHAPTER 4

Exercise 1: For each of the following statements determine whether it is true or false.
(1) $\langle A, B \mid A^5, B^2, (AB)^3 \rangle$ is a PCP.
(2) $\langle A, B \mid A^2, B^4, [A, B] = B^2 \rangle$ is a PCP.
(3) If G has a PCP then it is finite and soluble.

Exercise 2: If $G = \langle A, B \mid A^{26} = B^3 = 1, [A, B] = A^2 \rangle$ find the order of BA.

Exercise 3:
Let $G = \langle A, B, C \mid [A, B] = C, [A, C], [B, C] \rangle$.
Express $(BA)^3$ in the form $C^m B^n A^r$.

Exercise 4:
Let $G = \langle A, B, C \mid A^{10} = B^6 = C^8 = 1, [A, B] = C, [A, C] = [B, C] = 1 \rangle$.
(a) Show by induction that $B^{-n}AB = AC^n$ for all n.
(b) Find $|G|$; (c) Find $|Z(G)|$; (d) Find $|G'|$.

Exercise 5:
Let $G = \langle A, B, C \mid A^m = 1, B^n = A^k, C^d = B^e A^f,$

$$[A, B] = A^r, [A, C] = A^s, [B, C] = B^uA^v\rangle.$$

Show that $A^M = 1$ where

$$M = GCD(m, (r + 1)^{GCD(n, u)} – 1) \text{ and}$$

$$B^{MT} = 1 \text{ where } T = GCD(n, (u + 1)^d – 1).$$

SOLUTIONS FOR CHAPTER 2

Exercise 1:

(1) FALSE: A relator of the form $(AB)^3$ is not permitted in a power-commutator presentation. In fact this is a presentation for $\mathbf{A}_5$ which, being non-soluble, does not have a power-commutator presentation.

(2) FALSE: Technically it fails to be a power-commutator presentation because we have written the generators in the order A, B and so [A, B] would have to be a word in just A. But, of course, if we wrote the presentation as $\langle B, A \mid B^4, A^2, [B, A] = B^2\rangle$ it would qualify. The group being presented is $\mathbf{D}_8$.

(3) TRUE

Exercise 2: Since $[A, B] = A^2$, $BA = A^3B$.

Hence $(BA)^2 = BABA = B^2A^4$ and

$(BA)^3 = BAB^2A^4 = B^3A^9A^4 = B^3A^{13} = A^{13}$.

Hence $(BA)^6 = A^{26} = 1$, so BA has order 6.

AB always has the same order as BA and so it too has order 6.

Exercise 3: Since $[A, B] = C$, $AB = BAC$. Since C commutes with both A and B, $AB = CBA$.

So $(BA)^2 = B(AB)A = B(CBA)A = CB^2A^2$.

And $(BA)^3 = (BA)CB^2A^2 = CB(AB)BA^2$
$= CB(CBA)BA^2 = C^2B^2(AB)A^2$
$= C^2B^2(CBA)A^2$
$= C^3B^3A^3$.

Exercise 4: (a) Since $A^{-1}B^{-1}AB = C$, $B^{-1}AB = AC$, so it holds for $n = 1$.
Suppose $B^{-n}AB^n = AC^n$.
Then $B^{-1}(B^{-n}AB^n)B = B^{-1}AC^nB = ACC^n = AC^{n+1}$.
(b) Since $B^6 = 1$, $A = B^{-6}AB^6 = AC^6$ so $C^6 = 1$.
Since $C^8 = 1$ it follows that $C^2 = 1$.
(c) $|G| = 120$ since every element can be expressed uniquely as $A^mB^nC^r$ where:

$m = 0, 1, 2, \ldots, 9$, $n = 0, 1, 2, \ldots, 5$, $r = 0, 1$.

(d) Clearly $C \in Z(G)$. Also, $B^{-2}AB^2 = AC^2 = A$ so $B^2 \in Z(G)$. And $B^{-1}A^2B = (AC)^2 = A^2C^2$ so $A^2 \in Z(G)$. Hence $Z(G) = \{A^{2m}, B^{2n}, C\}$ which has order $5 \times 3 \times 2 = 30$.
(e) $G' = \langle C \rangle$ so $|G'| = 2$.

Exercise 5: Since $B^n = A^k$, it follows that B^n commutes with A.
Since $(BC)^{-1}A(BC) = A^{(r+1)(s+1)} = (CB)^{-1}A(CB)$ it follows that $[B, C] = (CB)^{-1}(BC)$ commutes with A.
Hence B^uA^v commutes with A, so B^v commutes with A.
It follows that $B^{\mathrm{GCD}(n, u)}$ commutes with A.
Conjugating A by $B^{\mathrm{GCD}(n, u)}$ gives $A^{(r+1)^{\mathrm{GCD}(n, u)}}$.
So $A^{(r+1)^{\mathrm{GCD}(n, u)}} = A$ and so

if $P = (r + 1)^{GCD(n, u)} - 1$, $A^P = 1$.
But $A^m = 1$ and so $A^{GCD(m, P)} = 1$.
Let $H = \langle A \rangle$. Then G/H has the presentation:

$$\langle B, C \mid B^n = 1, C^d = B^e, [B, C] = B^u \rangle$$

and so by the first part applied to G/H:

$$B^T \in H \text{ where } T = GCD(n, (u + 1)^d - 1).$$

Hence $B^{MT} = 1$.